GRADED BETTI NUMBERS OF CYCLE GRAPHS AND STANDARD YOUNG TABLEAUX

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Abstract. We give a bijective proof that the Betti numbers of a minimal free resolution of the Stanley-Reisner ring of a cycle graph (viewed as a one-dimensional simplicial complex) are given by the number of standard Young tableaux of a given shape.

1. Introduction

In a recent paper, Dochtermann [1] studied the (graded) Betti numbers $\beta_{i,j}(C_n)$ of a minimal free resolution of the Stanley-Reisner ring of the cycle graph C_n, viewed as a one-dimensional simplicial complex. He showed in [1, Theorem 4.3] that the nonzero Betti numbers of the resolution are $\beta_{0,0}(C_n) = \beta_{n-2,n}(C_n) = 1$ and

$$\beta_{j-1,j}(C_n) = \#\{\text{standard Young tableaux of shape } (j, 2, 1^{n-j-2})\} \tag{1}$$

for $2 \leq j \leq n-2$. Specifically, he showed that the left- and right-hand sides of Equation (1) satisfy a common recursion formula. In this note, we offer a bijective proof of this fact that preserves a natural duality present in each of the respective objects of interest.

2. Preliminaries

For the sake of brevity, we will adhere to the standard definitions and notation established in Miller and Sturmfels [2] and Stanley [3, 4], and we refer to these books for any undefined terms presented throughout this paper. We use the convention that Young tableaux are arranged in left-justified rows of weakly decreasing length in which the first (top) row is the longest and. For a standard Young tableau T, we denote by $T(i,j)$ the entry in the i^{th} row (from the top) and j^{th} column (from the left) in T.

For a simplicial complex Δ on vertex set V and $W \subseteq V$, we use $\Delta[W] := \{F \in \Delta \ : \ F \subseteq W\}$ to denote the *restriction* of Δ to the vertices in W, $\overline{W}$ to denote the set complement of W in V, and C_n to denote the standard cycle graph on n vertices, i.e., the graph on vertex set $[n] := \{1, 2, \ldots, n\}$ whose edge set consists of all pairs $\{i, j\}$ such that $i - j \equiv \pm 1 \mod n$.

We recall Hochster's formula, which will be the main tool in our analysis.

Theorem 2.1 (Hochster's formula)**.** *Let Δ be a simplicial complex on vertex set V, let $\mathbf{k}$ be a field, and let $\mathbf{k}[\Delta]$ be the Stanley-Reisner ring of Δ. Then the graded Betti numbers of a minimal free resolution of $\mathbf{k}[\Delta]$ are given by*

$$\beta_{i,j}(\Delta) = \sum_{W \in \binom{V}{j}} \dim_{\mathbf{k}} \widetilde{H}_{j-i-1}(\Delta[W]; \mathbf{k}). \tag{2}$$

When $\Delta = C_n$, it is clear from Equation (2) that $\beta_{0,0}(C_n) = 1$ and that $\beta_{n-2,n}(C_n) = 1$ by taking $W = \emptyset$ and $W = [n]$, respectively. Furthermore, the restriction of C_n to any proper, nonempty subset of vertices can only have non-vanishing homology in dimension 0, so the only remaining nonzero Betti numbers in the resolution of C_n are those $\beta_{j-1,j}(C_n)$ with $2 \leq j \leq n-2$. (If $j = 1$ or $j \geq n-1$, the restriction of C_n to any subset of j vertices is connected, and hence does not contribute to the sum in Equation (2).)

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3. The bijection

Our primary goal is to understand the combinatorics of $\beta_{j-1,j}(C_n)$ for $2 \leq j \leq n-2$. By Theorem 2.1, we know every subset $W \in \binom{[n]}{j}$ contributes one less than the number of connected components of $\Delta[W]$ to $\beta_{j-1,j}(C_n)$, so our initial aim will be to associate to every standard Young tableau of shape $(j, 2, 1^{n-j-2})$ a unique pair (W, X), where $W \in \binom{[n]}{j}$ and X represents a distinguished connected component of $C_n[W]$.

Definition 3.1. For $n \geq 4$ and $2 \leq j \leq n-2$, let $\mathscr{Y}(j,n)$ denote the set of standard Young tableaux of shape $(j, 2, 1^{n-j-2})$.

Since every standard Young tableau filled with the numbers in $[n]$ has a box labeled 1 in its upper left corner, the number 1 must be distinguished in terms of the restrictions $C_n[W]$ in any bijection under consideration. At the same time, for any proper, nonempty $W \subset [n]$, the restrictions $C_n[W]$ and $C_n[\overline{W}]$ have the same number of connected components, so any proposed bijection must somehow condition on the presence/absence of 1 in a set W and the connected components of the restrictions $\Delta[W]$ or $\Delta[\overline{W}]$, based on which of these sets contains vertex 1.

Definition 3.2. For every subset $W \subset V$, let

$$m(W) := \begin{cases} \{\min(X) \ : \ X \text{ is a connected component of } C_n[W]\} & \text{if } 1 \notin W, \\ \{\min(X) \ : \ X \text{ is a connected component of } C_n[\overline{W}]\} & \text{if } 1 \in W, \end{cases}$$

and $m'(W) := m(W) \setminus \min(m(W))$.

Since $C_n[W]$ and $C_n[\overline{W}]$ have the same number of connected components, it follows that $|m(W)|$ is equal to (and $|m'(W)|$ is one less than) the number of connected components of $\Delta[W]$. Note that the knowledge of $1 \in W$ and $a \in m(W)$ is sufficient to determine a component of $C_n[W]$ (or $C_n[\overline{W}]$ if $1 \notin W$).

Definition 3.3. For $n \geq 4$ and $2 \leq j \leq n-2$, let

$$\mathscr{S}(j,n) = \left\{ (W, a) \ : \ W \in \binom{[n]}{j} \text{ and } a \in m'(W) \right\}.$$

Remark 3.4. Observe that $m'(W)$ is implicitly required to be nonempty and hence $C_n[W]$ has at least two connected components for each W under consideration here.

At this point we are ready to present our bijection between $\mathscr{Y}(j,n)$ and $\mathscr{S}(j,n)$, but before we continue, let us first turn our attention to the set $\mathscr{Y}(j,n)$ for some brief motivation: If T is an element of $\mathscr{Y}(j,n)$, then the first row of T has j boxes filled by unique elements of $[n]$, the first column of T has $n-j$ boxes filled by unique elements of $[n]$, and, since T is standard, we know the element 1 must be located at position $T(1,1)$. Thus, for a given $W \in \binom{[n]}{j}$, it is natural to associate a standard Young tableau to W by first filling the first row of the table with the elements of W if $1 \in W$ and otherwise filling the first column of the table by the elements of $\overline{W}$ if $1 \notin W$. The set W is not sufficient to determine a single standard Young tableau under this rule, however, because $C_n[W]$ and $C_n[\overline{W}]$ may have many connected components. To account for the different components, we make use of the box at position $(2,2)$ of our Young diagram.

Definition 3.5. For every $T \in \mathscr{Y}(j,n)$, let $a_T := T(2,2)$, let B be the box in T that contains the number $a_T - 1$, and set

$$W_T := \begin{cases} \{T(1,i) \ : \ 1 \leq i \leq j\} & \text{if } B \text{ lies in the first row of } T, \\ \{T(2,2)\} \cup \{T(1,i) \ : \ 2 \leq i \leq j\} & \text{if } B \text{ lies in the first column of } T. \end{cases}$$

We now proceed with the main result of this paper:

Theorem 3.6. *For every $n \geq 4$ and $2 \leq j \leq n-2$, the function $\phi : \mathscr{Y}(j,n) \to \mathscr{S}(j,n)$ given by $\phi(T) = (W_T, a_T)$ is a bijection.*

Example 3.7. We exhibit $\phi : \mathscr{Y}(2,5) \to \mathscr{S}(2,5)$. The subsets $W \subseteq [5]$ for which $C_5[W]$ has multiple connected components correspond to the chords of C_5, so $\mathscr{S}(2,5)$ consists of the following ordered pairs:

$$(\{2,4\},4), \quad (\{2,5\},5), \quad (\{3,5\},5), \quad (\{1,3\},4), \quad (\{1,4\},5).$$

Moreover, $\mathscr{Y}(2,5)$ consists of the following five fillings of the shape $(2,2,1)$, which are shown below with their corresponding images under ϕ.

$$T_1 = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline 5 \\ \cline{1-1}\end{array} \qquad T_2 = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline 5 \\ \cline{1-1}\end{array} \qquad T_3 = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 5 \\ \hline 4 \\ \cline{1-1}\end{array} \qquad T_4 = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 5 \\ \hline 4 \\ \cline{1-1}\end{array} \qquad T_5 = \begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 & 5 \\ \hline 3 \\ \cline{1-1}\end{array}$$

$$\phi(T_1) = (\{2,4\},4) \qquad \phi(T_2) = (\{1,3\},4) \qquad \phi(T_3) = (\{2,5\},5) \qquad \phi(T_4) = (\{3,5\},5) \qquad \phi(T_5) = (\{1,4\},5).$$

Example 3.8. The case that $n = 6$ and $j = 3$ is the first case in which we can have a restricted subcomplex with more than two connected components. If $W = \{2,4,6\}$, then $m'(W) = \{4,6\}$ and the tableaux corresponding to $(\{2,4,6\},4)$ and $(\{2,4,6\},6)$, respectively, are

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 6 \\ \hline 3 & 4 \\ \cline{1-2} 5 \\ \cline{1-1}\end{array} \qquad \text{and} \qquad \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 & 6 \\ \cline{1-2} 5 \\ \cline{1-1}\end{array} \, .$$

Proof of Theorem 3.6. We begin showing that ϕ is injective: Suppose that T and T' are tableaux for which $\phi(T) = \phi(T')$. Let B be the box in T containing the number $a_T - 1$ and B' be the box in T' containing the number $a'_T - 1$. We consider two cases based on whether or not $1 \in W_T = W_{T'}$.

Case 1.1. Suppose $1 \in W_T = W_{T'}$. Then the entries of the first rows of T and T' are the elements of $W_T = W_{T'}$. Since T and T' are standard, these entries must be written in increasing order, so the first rows of T and T' must be the equal. Since $a_T = a_{T'}$, we also get that $T(2,2) = T'(2,2)$. Again, since T and T' are standard, it follows that the remaining entries, which must all lie in the respective first columns of T and T', are equal. Therefore, $T = T'$.

Case 1.2. Suppose $1 \notin W_T = W_{T'}$. Then the entries of the first columns of T and T' are the elements of the complement of $W_T = W_{T'}$ in $[n]$. Since T and T' are standard, these entries must be written in increasing order, so the first columns of T and T' must be equal. Since $a_T = a_{T'}$, we also get that $T(2,2) = T'(2,2)$. Again, since T and T' are standard, it follows that the remaining entries, which must all lie in the respective first rows of T and T', are equal. Therefore, $T = T'$.

Next, we show that ϕ is surjective: Let (W,a) be an element in $\mathscr{S}(j,n)$. We consider two cases based on whether or not $1 \in W$. Recall by our construction that $1 \in W$ if and only if $a \notin W$.

Case 2.1. Suppose $1 \in W$ and consider the tableau T of shape $(j, 2, 1^{n-j-2})$ filled in the following way:

- Sort W and fill it into the first row of T;
- Enter a in the $(2,2)$ position of T;
- Sort $\overline{W} - \{a\}$ and fill it into the rest of the first column of T.

It is clear that this filling is well-defined and that each element of $[n]$ belongs to one of the boxes of T. Let $b = T(1,2)$ and $c = T(2,1)$. To show that T is a standard filling, it suffices to prove that $a > b$ and $a > c$. Observe that b is the second-smallest element of W. If $b = 2$, then it is clear that $a > b$. Otherwise $2 \notin W$, which means $\{2, \ldots, b-1\}$ is a connected component of $C_n[\overline{W}]$, which implies that $\min(m(W)) = 2$. Thus, every element of $m'(W)$, in particular a, is greater than b, since the remaining connected components of

$C_n[\overline{W}]$ are subsets of $\{b+1\ldots,n\}$. This proves that $a>b$. To see that $a>c$, we recall that $a\notin W$ and, by construction, that a cannot be the smallest element of $\overline{W}$. It follows that c must be the smallest element of $\overline{W}$, and hence $a>c$. This establishes that T is standard.

Case 2.2. Suppose $1\notin W$ and consider the tableau T of shape $(j,2,1^{n-j-2})$ filled in the following way:

- Sort $\overline{W}$ and fill it into the first column of T;
- Enter a in the $(2,2)$ position of T;
- Sort $W-\{a\}$ and fill it into the rest of the first row of T.

Let $b=T(1,2)$ and $c=T(2,1)$ as before. To show that the filling of T is standard, it suffices to prove that $a>b$ and $a>c$. Observe that c is the second-smallest element of $\overline{W}$. If $c=2$, then it is clear that $a>c$. Otherwise $2\in W$, which means $\{2,\ldots,c-1\}$ is a connected component of $C_n[W]$, which implies that $\min(m(W))=2$. Thus, every element of $m'(W)$, in particular a, is greater than c. To see that $a>b$, we observe that b is the smallest element of W and hence $\min(m(W))=b$. Therefore, each element of $m'(W)$, in particular a, is greater than b. This establishes that T is standard.

In both Cases 2.1 and 2.2, it is clear from our definition that $\phi(T)=(W,a)$. □

Remark 3.9. We noted earlier that for any proper, nonempty subset $W\subset[n]$, the restrictions $C_n[W]$ and $C_n[\overline{W}]$ have the same number of connected components, which implies that $\beta_{j-1,j}(C_n)=\beta_{n-j-1,n-j}(C_n)$ for any $1\le j\le n-1$. This duality is expected since $\mathbf{k}[C_n]$ is known to be Gorenstein. The duality is reflected in the combinatorics on standard Young tableau in the form of transposition. (The transpose T^* of a standard Young tableaux T of shape $(j,2,1^{n-j-2})$ is a standard Young tableaux of shape $(n-j,2,1^{j-2})$.) The bijection defined in Theorem 3.6 establishes that these two notions of duality are compatible, that is $W_{T^*}=\overline{W}_T$ and $a_{T^*}=a_T$ for the transpose T^* of any tableau T.

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